Alan Turing's note to Rolf Noskwith on unit equilateral triangles in n dimensions

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This is Turing's note, with only the following changes: footnotes have been added, to clarify, specify necessary conditions, or spotlight when corrections are made; abbreviations have been expanded; limits have been provided for the indices; punctuation and the word 'the' have been added to facilitate reading; square brackets were added in one equation to clarify; and equation numbers have been introduced for Turing's two conditions and the specification of the question he sets out to answer.

We construct an equilateral [unit][a] triangle in n dimensions [in the first 2^n-tant][b] with 0^{th} vertex at the origin[c] and the i^{th} vertex having coordinates ℓ_{ij}, $\ell_{ij} = 0$ if $j > i$.[d] One easily verifies that we must have

$$\sum_{1 \le j \le i} \ell_{ij}^2 = 1 \qquad \forall\ 1 \le i \le n \tag{C1}$$

and

$$\sum_{i<k} \ell_{ij} = (k+1)\ell_{kj} \qquad \forall\ 2 \le k \le n \ \ and \quad j \ne k \tag{C2}$$

(the second condition means that the perpendicular from a vertex to the opposite side meets it in the centre of gravity of the vertices of that side). These conditions determine the ℓ_{ij} completely[e]. In fact:[f]

$$\begin{aligned} \ell_{ii} &= \sqrt{\frac{i+1}{2i}} && \forall\ 1 \le i \le n \\ \ell_{ij} &= \frac{1}{\sqrt{2j(j+1)}} && \forall\ j < i \\ (\ell_{ij} &= 0 && \forall\ j > i) \end{aligned}$$

which may be verified to satisfy the conditions.[g] Now we want to know how many sets of integers m_j[h] there are such that

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{n} \ell_{ij} m_i \right)^2 = 1 \tag{C3}$$

Now there are no solutions except with $m_n = 0, 1$ or -1, for if $m_n^2 \ge 4$

$$\sum_j \left(\sum_i \ell_{ij} m_i \right)^2 \ge \left(\sum_i \ell_{in} m_i \right)^2 = \ell_{nn}^2 m_n^2 \ge 4\ell_{nn}^2 = \frac{2(n+1)}{n} > 1$$

[a] Condition C1 stipulates this.

[b] the first quadrant, octant... This is not required for conditions C1 and C2, and the question involving condition C3 can be asked for any equilateral unit triangle in n dimensions, but the definitions Turing later gives for the ℓ_{ij} are all non-negative.

[c] Turing wrote 'first vertex at the origin' but the indexing in the rest of the paper begins at $i = 1$, $\ell_{11} = 1$ and so the origin is a 0^{th} vertex.

[d] The i^{th} vertex is $(\ell_{i1}, \ell_{i2}, \dots, \ell_{ii}, 0, \dots, 0)$ with $\ell_{ij} \ge 0$. Such an equilateral unit n-triangle is constructed by including the origin, taking the first vertex as $(1, 0, \dots, 0)$ and then, for each $k = 2, \dots, n$ adding the k^{th} vertex on the perpendicular line orthogonal to, and through the midpoint or 'average' of, the triangle that is the previous $k-1$ vertices plus the origin. Take for the k^{th} vertex the point on that line at distance 1 from the origin, and with positive k^{th} coordinate.

[e] if we constrain the n-triangle to the positive 2^n-tant

[f] The rest of the note takes these three equations as the definitions of the ℓ_{ij}. Turing wrote $\frac{1}{\sqrt{2j(j-1)}}$ for $j < i$ but later used the correct $\frac{1}{\sqrt{2j(j+1)}}$ and so we give the correct version here. This error of Turing's might suggest this document is a fair copy of his workings which don't survive.

[g] C1 and C2

[h] i.e., how many vectors $\mathbf{m} = (m_1, m_2, \dots, m_n)$. Later Turing introduces the notation g_n for the number of such vectors.

[i]Now suppose $m_n = 0$. The number of solutions is the number of the last lower dimension, g_{n-1} say.

Next suppose $m_n = 1$. Then $m_{n-1} = 0$ or -1, for

$$1 = \sum_j \left(\sum_i \ell_{ij} m_i \right)^2 \geq m_n^2 \ell_{nn}^2 + (\ell_{n,n-1} m_n + \ell_{n-1,n-1} m_{n-1})^2$$
$$= \frac{(n+1)}{2n} + \left(\frac{1}{\sqrt{2n(n-1)}} + \sqrt{\frac{n}{2(n-1)}} m_{n-1} \right)^2$$

i.e.[j]

$$\frac{n-1}{2n} \geq \frac{n}{2(n-1)} \left(m_{n-1} + \frac{1}{n} \right)^2 \tag{B}$$

Hence $-1 + \frac{1}{n} \leq m_{n-1} + \frac{1}{n} < 1$[k] i.e. $m_{n-1} = 0$ or -1.

Now suppose $m_{n-1} = 0$. Then the m_i with $i \leq n-2$ are required to satisfy the condition

$$\sum_{j \leq n-2} \left(\left[\sum_{i \leq n-2} \ell_{ij} m_i \right] + \frac{1}{\sqrt{2j(j+1)}} \right)^2 = \frac{n-1}{2n} - \frac{1}{2n(n-1)}$$
$$= \frac{(n-1)^2 - 1}{2n(n-1)}$$
$$= \frac{n-2}{2n-2}$$

Now $\frac{1}{\sqrt{2j(j+1)}}$ represents the coordinates of the centre of gravity of the vertices 1 to $n-2$, and $\frac{n-2}{2n-2}$ is the square of the distance of these vertices from that centre of gravity. Each vertex therefore represents a solution, giving us $n-1$ solutions with $m_n = 1$, $m_{n-1} = 0$.

Now suppose $m_{n-1} = -1$. Then we find that $\sum_i \ell_{ij} m_i = 0$ if $j \leq n-2$ (since the inequality B is only just satisfiable). This results in $m_i = 0$ for $i \leq n-2$[l,m] i.e. only one solution. Hence there are n solutions with $m_n = 1$. Similarly n with $m_n = -1$.[n]

$$g_n = g_{n-1} + 2n, \qquad g_1 = 2$$

Hence $g_n = n(n+1)$.

revised 2023-05-06: 'coordinates of the centre of gravity' for 'condition of the centre of gravity' typeset in LaTeX

[i]Turing is going to prove that for $n \geq 2$, $g_n = g_{n-1} + 2n$. $g_1 = 2$ because the solutions to condition C3 for $n = 1$ are ($\ell_{11} = 1$) $m_1 = \pm 1$. It is readily verified that for $n = 2$, $\mathbf{m} = \pm(1,0), \pm(0,1), \pm(1,-1)$ are the only solutions to C3, thus $g_2 = g_1 + 2 * 2$ is true. So from here on we can assume $n \geq 3$.

[j]There is no equation A, further suggesting that Turing was 'fair-copying' his notes as a draft-y outline of a proof, for Rolf.

[k]Turing forgot the '$-1 + \frac{1}{n} \leq$' or else he left off the absolute value marks.

[l]Turing wrote '$j \leq n-2$'.

[m]This follows by a reverse-induction argument; start with $j = n-2$ and work backwards to $j = 1$.

[n]Following the above argument with $m_n = -1$ leads to the only possibilities being $m_{n-1} = 1$ (providing one solution, mostly zeroes) and $m_{n-1} = 0$ (providing $n-1$ solutions), but that is unnecessary because $\mathbf{m}$ is a solution if and only if $-\mathbf{m}$ is a solution and so there are 2n solutions with $m_n \neq 0$.